A quantization of Sylvester's law of inertia

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(joint work in progress with S. Moore)

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Overview

Setting

$$\text{Real algebraic space } X$$

$$\text{(Liberation, } q\text{-deformation, ...)} \downharpoonleft\upharpoonright (xy = yx, q \to 1, \ldots)$$

$$\text{NC } *\text{-algebra } \mathcal{O}(\mathbb{X}) + \text{ Spectral conditions}$$

$$\text{Representation category} \downharpoonleft\upharpoonright \text{Forgetful functor}$$

$$W^* - \text{category } \mathbf{X}.$$

Self-adjoint matrices

Fix $N \in \mathbb{Z}_{\geqslant 1}$. We put

$$H(N) = \{h \in M_N(\mathbb{C}) \mid h^* = h\}.$$

Sylvester's law of inertia

Spectral theorem

Let Λ be the set of multi-sets of N real numbers,

$$\Lambda = \mathbb{R}^N / \mathrm{Sym}(N).$$

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Theorem (Spectral theorem (Cauchy))

Consider

$$\mathrm{Ad} : H(N) \times U(N) \to H(N), \qquad \mathrm{Ad}_u(h) = u^* h u.$$

Then

$$H(N)/U(N) \overset{\lambda}{\cong} \Lambda, \qquad [h] \mapsto \textit{eigenvalues of } h.$$

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Note: we have a natural section

$$\Lambda \to H(N), \qquad \lambda = \{\lambda_1 \geqslant \ldots \geqslant \lambda_N\} \mapsto \mathrm{diag}(\lambda_1, \ldots, \lambda_N).$$

Sylvester's law of inertia

Sylvester's law of inertia

Let Sign be the set of multi-sets of N signs,

$$\mathrm{Sign} = \{-1, 0, 1\}^N / \mathrm{Sym}(N).$$

We can identify Sign with partitions $N = N_0 + N_+ + N_-$.

Sylvester's law of inertia

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Theorem (Sylvester's law of inertia)

Consider

$$\mathrm{Ad} : H(N) \times GL(N, \mathbb{C}) \to H(N), \qquad \mathrm{Ad}_x(h) = x^* h x.$$

Then

$$H(N)/GL(N, \mathbb{C}) \overset{S}{\cong} \mathrm{Sign}, \qquad [h] \mapsto \textit{signs of eigenvalues of } h.$$

Sylvester's law of inertia

Conjugation by the triangular subgroup

Let $T(N)$ be the uppertriangular matrices with positive diagonal

$$T(N) = \{t \mid t_{ii} > 0 \text{ and } t_{ij} = 0 \text{ for } i > j\}.$$

Then we have the Gauss decomposition

$$U(N) \times T(N) \cong GL(N, \mathbb{C}), \qquad (u, t) \mapsto ut.$$

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Let $\mathcal{S}$ be the set of shapes

$$\mathcal{S} = \{S \in H(N) \mid Se_k = u_k e_{\sigma(k)} \text{ with } |u_k|^2 = |u_k|\}.$$

Sylvester's law of inertia

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Theorem (Elsner '79, Gohberg-Goldberg '82)

Consider

$$\mathrm{Ad} : H(N) \times T(N) \to H(N), \qquad \mathrm{Ad}_t(h) = t^* ht.$$

Then

$$\mathcal{S} \overset{\rho}{\cong} H(N)/T(N), \qquad S \mapsto [S].$$

Sylvester's law of inertia

Intersection of orbits

Let $C_h = \mathrm{Ad}_{U(N)}(h) \cap \mathrm{Ad}_{T(N)}(h)$, and put

$$\widehat{H}(N) = \{C_h\}.$$

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Theorem

The C_h are connected, and we have an isomorphism

$$\begin{array}{ccccc} & & \widehat{H}(N) & & \\ & \swarrow & & \searrow & \\ H(N)/T(N) & & & & H(N)/U(N) \\ & \searrow & & \swarrow & \\ & & H(N)/GL(N,\mathbb{C}) & & \end{array} \quad \cong \quad \begin{array}{ccccc} & & \mathcal{S} \times_{\mathrm{Sign}} \Lambda & & \\ & \swarrow & & \searrow & \\ \mathcal{S} & & & & \Lambda \\ & \searrow & & \swarrow & \\ & & \mathrm{Sign} & & \end{array}$$

Sylvester's law of inertia

Example

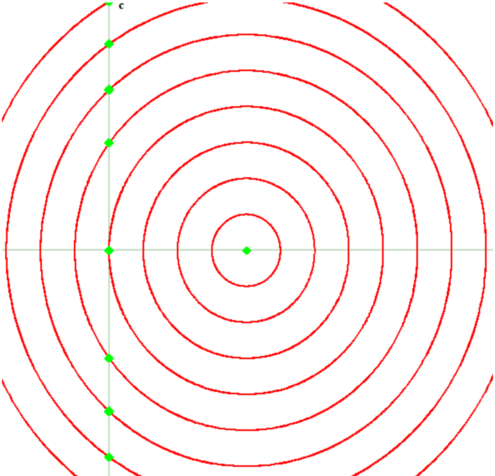
The set $\widehat{H}(2)$ has the following representatives h:

- Signature $(0,0)$: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
- Signature $(0,\pm 1)$: $\begin{pmatrix} \pm\lambda & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & \pm\lambda \end{pmatrix}$,
- Signature $(\pm 1,\pm 1)$: $\begin{pmatrix} \pm\lambda_1 & 0 \\ 0 & \pm\lambda_2 \end{pmatrix}$,
- Signature $(\pm 1,\mp 1)$: $\begin{pmatrix} \pm\lambda_1 & 0 \\ 0 & \mp\lambda_2 \end{pmatrix}$, $\begin{pmatrix} 0 & \bar{u}\lambda \\ u\lambda & \delta \end{pmatrix}$

Sylvester's law of inertia

Visual representation

Matrices $\begin{pmatrix} a & c \\ c & t-a \end{pmatrix}$ with $t \in \mathbb{R}$ fixed and $a, c \in \mathbb{R}$:



The semiclassical picture

Poisson manifolds

Definition

M is a Poisson manifold if we have a Lie bracket

$$\{-,-\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

with

$$\{f,-\} \in \mathrm{Der}(C^\infty(M)), \qquad \forall f \in C^\infty(M).$$

Hence $\{f,-\}$ gives a vector field X_f with $X_{\{f,g\}} = [X_f, X_g]$.

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Definition

The symplectic leaves of $(M, \{-,-\})$ are the leaves of the foliation associated to the involutive distribution $\mathscr{P}$ with

$$\mathscr{P}_m = \{(X_f)_m \mid f \in C^\infty(M)\}.$$

The semiclassical picture

$H(N)$ as Poisson manifold

If $(M, \{-,-\})$ is Poisson, we have a bivector field

$$\mathscr{X} \in \Lambda^2 TM, \qquad \{f, g\}(m) = (df_m \otimes dg_m, \mathscr{X}_m).$$

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Theorem (Karolinksy '95, Lu-Yakimov '08)

Let

$$r = \sum_i e_{ii} \otimes e_{ii} + 2\sum_{i<j} e_{ij} \otimes e_{ji}, \quad r' = \sum_i e_{ii} \otimes e_{ii} + 2\sum_{i<j} e_{ji} \otimes e_{ij}$$

Then $H(N)$ is a Poisson manifold through the bivector field

$$i\mathfrak{X}_h = r'(h \otimes h) - (h \otimes h)r + (h \otimes 1)r(1 \otimes h) - (1 \otimes h)r'(h \otimes 1)$$

Moreover, $\widehat{H}(N)$ is the partition of $H(N)$ into its symplectic leaves.

Kontsevich formality

Theorem (Kontsevich formality)

Let $(M, \{-,-\})$ be a Poisson manifold. Then there exists an (essentially unique) associative product $*$ on $C^\infty(M)[[h]]$ with

$$f * g = fg + ih\{f, g\} + O(h^2).$$

We call $(C^\infty(M)[[h]], *)$ a (formal) deformation quantization.

The semiclassical picture

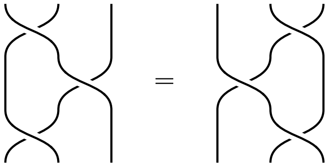
Braid operator

Let $q_h = e^h \in \mathbb{C}[[h]]$. We have the braid operator

$$\hat{R} = \sum_{ij} q_h^{-\delta_{ij}} e_{ji} \otimes e_{ij} + (q_h^{-1} - q_h) \sum_{i<j} e_{jj} \otimes e_{ii}$$
$$\in M_N(\mathbb{C}[[h]]) \otimes M_N(\mathbb{C}[[h]]).$$

It satisfies the braid relation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$$



The semiclassical picture

REA

Definition

Reflection Equation Algebra A_R over $\mathbb{C}[[h]]$:

- Generators: Z_{ij} for $1 \leqslant i, j \leqslant N$.
- Relations: with $Z = (Z_{ij}) \in M_N(A_R)$ we ask

$$\hat{R}(1 \otimes Z)\hat{R}(1 \otimes Z) = (1 \otimes Z)\hat{R}(1 \otimes Z)\hat{R}$$

The semiclassical picture
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Theorem
There exists a deformation quantization for $(H(N), \{-,-\})$ *with*

$$A_R \hookrightarrow C^\infty(H(N))[[h]].$$

The semiclassical picture
Specialisation

Denote $A_R = \mathcal{O}_h(H(N))$. Then we have a $\mathbb{C}[q_h, q_h^{-1}]$-subalgebra

$$\mathcal{O}_h(H(N)) \supseteq \mathcal{O}_{q_h}(H(N)) = \langle q_h^{\pm 1}, Z_{ij} \rangle.$$

Moreover, for $q \in \mathbb{C} \backslash \{0\}$ we can specialize to a $\mathbb{C}$-algebra

$$\mathcal{O}_q(H(N)) = \mathcal{O}_{q_h}(H(N))/(q_h - q).$$

The semiclassical picture

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Definition

Let $0 < q < 1$. We define the $*$-REA $\mathcal{O}_q(H(N))$ by the $*$-operation

$$Z^* = Z.$$

The semiclassical picture

Example: $N = 2$

For $N = 2$, we have $Z = \begin{pmatrix} z & w \\ v & u \end{pmatrix}$ selfadjoint and

$$zw = q^2 wz, \qquad zv = q^{-2} vz, \qquad zu = uz,$$

$$vw = wv + (1 - q^2) z^2 - (1 - q^2) uz,$$

$$vu = q^2 zv + uv - q^4 zv.$$

Goal and first main result

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Classify the irreducible bounded $*$-representations of $\mathcal{O}_q(H(N))$.

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$$\mathbf{H}_q(N) = \{\text{bounded } * - \text{representations of } \mathcal{O}_q(H(N))\}.$$

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Theorem (DC-Moore (2020))

The $*$-algebra $\mathcal{O}_q(H(N))$ is C-faithful and type I:*

- $\cap_{\pi \in \mathbf{H}_q(N)} \mathrm{Ker}(\pi) = \{0\}$.
- *Each* $C^*(\pi) := \pi(\mathcal{O}_q(H(N)))^{norm\text{-}cl}$ *is type I.*

Center of $\mathcal{O}_q(H(N))$

Theorem (Nazarov-Tarasov '94, Pyatov-Saponov '95)

The center $\mathscr{Z}(O_q(H(N))) = \mathbb{C}[\sigma_1, \ldots, \sigma_N]$ *with*

$$Z^N - \sigma_1 Z^{N-1} + \ldots + (-1)\sigma_N = 0.$$

If π is an irreducible bounded $*$-representation of $\mathcal{O}_q(H(N))$, then

$$\pi(\sigma_k) = \sigma_k^\pi \in \mathbb{R}.$$

Second main result

For $s \in \mathbb{R}^N$ we write $p_s(x) = x^N - s_1 x^{N-1} + \ldots + (-1)s_N$.

Theorem (DC-Moore 2020)

Let $s \in \mathbb{R}^N$. There exists an irreducible $*$-representation with $s_k = \sigma_k^\pi$ if and only if the roots of p_s are of the form

$$\lambda_\pi = \{\underbrace{0, \ldots, 0}_{N_0}, q^{2\alpha+2m_1}, \ldots, q^{2\alpha+2m_{N_+}}, -q^{2\beta+2n_1}, \ldots, -q^{2\beta+2n_{N_-}}\}$$

where $m_i \neq m_j$ and $n_i \neq n_j$ are integers.

Statement of the main results
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Definition

We call $(N_0, N_+, N_-, \beta - \alpha + \mathbb{Z})$ the extended signature of π.

The quantum algebra of $GL(N, \mathbb{C})$

Definition

The $*$-algebra $\mathcal{O}_q^{\mathbb{R}}(GL(N, \mathbb{C}))$ is generated by $X_{ij}, X'_{ij}, Y_{ij}, Y'_{ij}$ with

$$X^* = Y' = Y^{-1} \qquad Y^* = X' = X^{-1}$$

$$\hat{R}X_1X_2 = X_1X_2\hat{R} \qquad \hat{R}Y_1Y_2 = Y_1Y_2\hat{R} \qquad \hat{R}X_1Y_2 = Y_1X_2\hat{R}$$

The quantum group associated to $GL(N, \mathbb{C})$

We have a W*-category

$$\mathbf{GL}_q(N) = \{\text{bounded } * -\text{representations of } \mathcal{O}_q^{\mathbb{R}}(GL(N))\}.$$

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As $\mathcal{O}_q^{\mathbb{R}}(GL(N, \mathbb{C}))$ is a Hopf $*$-algebra by

$$\Delta(X) = X_1 X_2,$$

we obtain a monoidal W*-category:

$$\mathbf{GL}_q(N, \mathbb{C}) \times \mathbf{GL}_q(N, \mathbb{C}) \to \mathbf{GL}_q(N, \mathbb{C}),$$

$$(\lambda_1, \lambda_2) \mapsto \lambda_1 \lambda_2 = (\lambda_1 \otimes \lambda_2) \circ \Delta.$$

Statement of the main results

Adjoint action

We have a coaction

$$\mathrm{Ad}_q : \mathcal{O}_q(H(N)) \to \mathcal{O}_q(H(N)) \otimes \mathcal{O}_q^{\mathbb{R}}(GL(N,\mathbb{C})), \qquad Z \mapsto X^* Z X.$$

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Hence $\mathbf{H}_q(N)$ becomes a module W*-category over $\mathbf{GL}_q(N,\mathbb{C})$ via

$$\mathbf{Ad} : \mathbf{H}_q(N) \times \mathbf{GL}_q(N,\mathbb{C}) \to \mathbf{H}_q(N),$$

$$(\pi, \lambda) \mapsto \mathrm{Ad}_\lambda(\pi) = (\pi \otimes \lambda) \circ \mathrm{Ad}_q\,.$$

Statement of the main results

The third main result

For $\pi_1, \pi_2 \in \mathbf{H}_q(N)$, we write

$$\pi_1 \preccurlyeq \pi_2 \qquad \text{iff} \qquad C^*(\pi_2) \twoheadrightarrow C^*(\pi_1).$$

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Theorem (Quantized Sylvester's law of inertia)

Two irreducible $*$-representations π_1, π_2 of $\mathcal{O}_q(H(N))$ have the same extended signature if and only if there is $\lambda \in \mathbf{GL}_q(N, \mathbb{C})$ with

$$\pi_1 \preccurlyeq \mathrm{Ad}_\lambda(\pi_2).$$

Statement of the main results

A conjecture

Let $\Lambda_q \subseteq \Lambda$ be the multi-sets of the form

$$\lambda = \{\underbrace{0, \ldots, 0}_{N_0}, q^{2\alpha+2m_1}, \ldots, q^{2\alpha+2m_{N_+}}, -q^{2\beta+2n_1}, \ldots, -q^{2\beta+2n_{N_-}}\}$$

To any irreducible π one can associate a 'generalized shape' S_π.

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To any irreducible π one can associate a 'generalized shape' S_π.

Based on Kirillov's orbit method, we make the following conjecture.

Conjecture

Let $\widehat{H}_q(N) = \{\text{irreducible } * \text{-representations of } \mathcal{O}_q(H(N))\}/\cong$. Then the following is a well-defined bijection:

$$\widehat{H}_q(N) \to \Lambda_q \times_{\mathrm{Sign}} \mathcal{S}, \quad \pi \mapsto (\lambda_\pi, S_\pi).$$